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17-36 - 1813
Great French analyst)

Lagrange's Methed :→

The general solution of linear partial differential equation

$$Pp + Qq = R \quad \text{is}$$

$f(u,v) = 0$, Where f is an arbitrary function.

& $u = u(x,y,z) = c_1$, $v = v(x,y,z) = c_2$ are the solutions of Legrang's auxilary eqn. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Formation of Lagrang's Auxilary equation:-

Standered form of first order (linear) Partial differential eqn is

$$Pp + Qq = R \quad — (1)$$

Let us two integrals be.

$$u = u(x,y,z) = c_1$$

$$\& \; v = v(x,y,z) = c_2$$

Differenceating $u = c_1$, we get

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

Differenceating $v = c_2$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} \cdot dz = 0$$

solving for dx, dy, dz, we get

$$\frac{dx}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z}}$$

or, $$\frac{dx}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{dy}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{dz}{\frac{\partial(u,v)}{\partial(x,y)}}$$

or, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, which are called Legrange's auxiliary (or subsidary) equation.

Working procedure to solve $Pp+Qq=R$

(1) change the partial differential equation in the form $Pp+Qq=R$.

(2) Form the Lagrange's auxilarly eq^n

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(3) Find two integrals from

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ as

$u=c_1$ & $v=c_2$

(4) The general integral of this equation $Pp+Qq=R$ is obtained by $f(u,v)=0$, where f is an arbitrary function.

Examples:-

(1) Solve $(y^2+\cancel{z^2}-x^2)\,p - 2xyq + 2zx = 0$

Sol^n:→ This equation is in the form of

$(y^2+z^2-x^2)\,p - 2xyq = -2zx$

$\therefore \; P = y^2+z^2-x^2$

$Q = -2xy, \qquad R = -2zx$

$\therefore$ Lagrange's auxilarly eq^n are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2zx}$$

From last ~~to~~ eq^n

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\therefore \log y = \log z + \log c_1$$

$$\Rightarrow \frac{y}{z} = c_1$$

Again using x, y, z as multiplies. we get

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2zx} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\therefore \log y = \log(x^2+y^2+z^2) - \log c_2$$

$$\therefore \frac{x^2+y^2+z^2}{y} = c_2$$

so general sol^n is

$$x^2+y^2+z^2 = y \cdot f\left(\frac{y}{z}\right)$$

or, $$\phi\left(\frac{x^2+y^2+z^2}{y}, \frac{y}{z}\right) = 0$$

(2) $zxp + yzq = xy$

sol^n:→ It is in the form $Pp + Qq = R$
so Lagrange's A·E is/are

$$\frac{dx}{zx} = \frac{dy}{yz} = \frac{dz}{xy}$$

From fst equations,

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \log y = \log x + \log c_1$$

$$\Rightarrow \frac{y}{x} = c_1$$

using $y, x, -2z$ as multipliers we get

$$\frac{dx}{zx} = \frac{dy}{yz} = \frac{dz}{xy} = \frac{y\,dx + x\,dy - 2z\,dz}{0}$$

$$\therefore \quad y\,dx + x\,dy - 2z\,dz = 0$$

$$\text{or,} \quad d(x\cdot y) - 2z\,dz = 0$$

$$\therefore xy = z^2 + c_2$$

So general sol^n is

$$xy - z^2 = f\left(\frac{y}{x}\right)$$

$$\text{or,} \quad \phi\left(xy - z^2, \frac{y}{x}\right) = 0$$

(3) $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$

→ It is of the form $Pp + Qq = R$

so. Lagrange's auxilarly equation is are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

using $\underline{x, y, z}$ as multipliers. we get

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \quad x^2 + y^2 + z^2 = c_1$$

From last equations, $\frac{dy}{y+z} = \frac{dz}{y-z}$

$$\therefore \quad y\,dy - z\,dy = y\,dz + z\,dz$$

$$\therefore \quad 2y\,dy = 2\,d(y.z) + 2z\,dz$$

$$\therefore \quad 2yz + z^2 - y^2 = c_2$$

so general solⁿ is

$$\phi\left[2yz + z^2 - y^2,\ x^2 + y^2 + z^2\right] = 0$$

(Q) $(mz - ny)p + (nx - lz)q = ly - mx$

solⁿ:→ Its Lagrange's A.E is

$$\frac{dx}{mz - ny} = \frac{dy}{nx - ly} = \frac{dz}{ly - mx}$$

using $\underline{l, m, n}$ as multipliers, we get

$$\therefore \text{each term} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

integrating, $lx+my+nz=c_1$

Again using x, y, z as multipliers, we get

$\therefore$ each term $= \dfrac{x\,dx+y\,dy+z\,dz}{0}$

$\therefore$ integrating, $x^2+y^2+z^2=c_2$

so general solⁿ is $\phi\,(x^2+y^2+z^2,\ lx+my+nz]=0$

(85) $(x^2-yz)p+(y^2-zx)q=z^2-xy$

sol: Lagrange's A.E is

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

$$\therefore \frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}=\frac{dz-dx}{(z-x)(x+y+z)}$$

$$\Rightarrow \frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

From 1st equations.

$$\frac{x-y}{y-z}=c_1$$

From last two, $\dfrac{y-z}{z-x}=c_2$

so general solⁿ is

$$\phi\left[\frac{x-y}{y-z},\ \frac{y-z}{z-x}\right]=0$$

(86) $y^2 z\, p + x^2 z\, q = xy^2$

sol^n:→ Lagrange's A.E is

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

From 1st & last we get

$$\frac{dx}{z} = \frac{dz}{x}$$

$$\therefore x\,dx - z\,dz = 0 \quad \therefore x^2 - y^2 = c_1$$

From 1st equations.

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

$$\therefore x^2 dx = y^2 dy$$

$$\therefore x^3 - y^3 = c_2$$

so general sol^n is $\phi\left[x^3 - y^3,\ x^2 - z^2\right] = 0$

(87) $z - xp - yq = a\sqrt{x^2+y^2+z^2}$

sol^n:→ This equation can be written as in Lagrange's form

$$xp + yq = z - a\sqrt{x^2+y^2+z^2}$$

so that Lagrange's A.E is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}}$$

From 1st two equations.

$$\log x = \log y + \log c_1$$

$$\therefore \frac{x}{y} = c_1$$

using x, y, z as multipliers we get

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}} = \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2 - az\sqrt{x^2+y^2+z^2}}$$

put $x^2+y^2+z^2 = \theta^2$

$\therefore x\,dx + y\,dy + z\,dz = \theta\,d\theta$

$\therefore$ above eqn is, $\dfrac{dz}{z - a\theta} = \dfrac{\theta\,d\theta}{\theta^2 - a\theta z}$

or, $\dfrac{dz}{z - a\theta} = \dfrac{d\theta}{\theta - az} = \dfrac{dz + d\theta}{(1-a)(z+\theta)}$

$= \dfrac{dx}{x} = \dfrac{dy}{y}$

$\left(\frac{1}{1-a}\right)\log(z+\theta) = \log c_2 x$

$\therefore (z+\theta) = (c_2 x)^{(1-a)}$

$\therefore$ general soln is

$$\sqrt{x^2+y^2+z^2} + z = \left[f\left(\frac{x}{y}\right)\cdot x\right]^{(1-a)}$$

(58) solve $p\cos(x+y) + q\sin(x+y) = z$

Soln:→ This is lagrange's form so its A.E are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$

Here, $\dfrac{dx+dy}{\cos(x+y)+\sin(x+y)} = \dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$

or, $\dfrac{[-\sin(x+y)+\cos(x+y)]\,d(x+y)}{\cos(x+y)+\sin(x+y)} = dx-dy$

integrating.

$$\log[\cos(x+y)+\sin(x+y)] = (x-y)+\log c_1$$

$$\therefore \cos(x+y)+\sin(x+y) = c_1 e^{x-y}$$

Also,

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z}$$

$$\therefore \frac{\frac{1}{\sqrt{2}}(dx+dy)}{\sin(x+y+\frac{\pi}{4})} = \frac{dz}{z}$$

$$\therefore \frac{d(x+y+\frac{\pi}{4})}{\sin(x+y+\frac{\pi}{4})} = \sqrt{2}\,\frac{dz}{z}$$

$$\therefore \log\tan\frac{4(x+y)+\pi}{8} = \sqrt{2}\log z + \log c_2$$

$$= \log z^{\sqrt{2}}\cdot c_2$$

$$\therefore \tan\frac{4(x+y)+\pi}{8} = z^{\sqrt{2}} c_2$$

So general sol^n is

$$\tan\frac{4(x+y)+\pi}{8} = z^{\sqrt{2}}\,\phi\left(\frac{\cos(x+y)+\sin(x+y)}{e^{x-y}}\right)$$

(9) solve $zxp - yzq = y^2 - x^2$

Solⁿ :→ It is in the Lagrange's form so its A.E is

$$\frac{dx}{zx} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2}$$

or, $$\frac{dx - dy}{z(x+y)} = \frac{dz}{(y+x)(y-x)}$$

$$\therefore (x-y)\, d(x-y) + z\,dz = 0$$

$$\therefore (x-y)^2 + z^2 = c_1$$

Again using x, y, z as multipliers we get

$$\frac{dx}{zx} = \frac{dy}{-zx} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore x^2 + y^2 + z^2 = c_2$$

so general solⁿ is $\phi\,[x^2+y^2+z^2,\ (x-y)^2+z^2] = 0$

(8) $$\frac{y-z}{yz}\, p + \left(\frac{z-x}{zx}\right) q = \frac{x-y}{xy}$$

⇒ multiplying both sides by $x \cdot y \cdot z$.

⟹ $x(y-z)p + y(z-x)q = z(x-y)$ is

the form $Pp + Qq = R$

so Lagrange's A.E is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

using 1, 1, 1 as multipliers, we get

$\therefore$ each term $= \dfrac{\cancel{x}dx + \cancel{y}dy + dz}{0}$

$\therefore \ dx + dy + dz = 0$

$\therefore \ x + y + z = c_1$

Again, using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers,

we get.

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\therefore \ xyz = c_2$$

$\therefore$ general sol^n is

$$\phi(x, y, z;\ x+y+z) = 0$$